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An Analytic Solution for Density Distribution in a Planetary Exosphere

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ABSTRACT

An analytical expression $\rho(r)=N_0[e^{(1-R/r)E}-(1-R^2/r^2)e^{-(rE/r+R)}]$, where E is a temperature dependent parameter and R is the radius of the base of exosphere, is derived for the density distribution in a planetary exosphere. The difference between this distribution and the barometric (Boltzmann) formula is small near the base of the exosphere but becomes significant at large r; at $r=\infty$ the barometric formula gives a finite density where our distribution tends to zero. It is shown that according to a strict collisionless exosphere model the particles in the velocity space are confined in a region bounded by a hyperbola and a quarter circle. Outside this region there are no particles; inside, they are distributed by a Maxwellian law. The physical significance of this difference and its effect on the escape rate are discussed.

Recently Öpik and Singer (1959, 1961) have presented a theory that gives the density distribution in a planetary exosphere. They assume a Maxwellian velocity distribution for the base of the exosphere which ejects particles into the exosphere. They then calculate the density distribution numerically by classifying the constituents of the neutral exosphere into essentially two components:

1.) The ballistic component, consisting of molecules ejected from the base of the exosphere.
2.) The orbiting component, consisting of molecules circling the planet in elliptic orbits not intersecting the base of the exosphere.

We wish to present an analytic expression for the ballistic density distribution, which has two advantages: 1) It can be compared directly with the barometric formula and hence gives more clearly a physical understanding of the Öpik-Singer theory. 2) In a more complicated problem like that of the thermoionic and photoelectric screening of bodies in space, where the potential has to be obtained from a Poisson equation, an analytic form for the charge distribution is necessary.

To calculate the ballistic density $\rho_T(r)$, we assume a Maxwellian velocity distribution at the base of the exosphere. The flux of molecules ejected from the base of the exosphere (at radius R) with velocity in the range V_p to V_p+dV_p and V_t to V_t+dV_t will be

$$8\pi^2R^2\left(\frac{M}{2\pi kT}\right)^{3/2}N_0e^{-(M/2kT)(V_p^2+V_t^2)}V_pV_t\,dV_p\,dV_t. \quad (1)$$

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Here M is the mass of the molecule, T the temperature, N_0 the number density, V_p the velocity component parallel to the radius vector, V_t the velocity component perpendicular to the radius vector.

Let $\mathbf{u}$ be the velocity of a particle at $\mathbf{r}$ $(r>R)$ which was ejected from the base of exosphere at velocity $\mathbf{v}$. Then by conservation of angular momentum and energy, we have

$$u_tr=V_tR, \quad (2)$$

$$u_t^2+u_p^2+\frac{2\phi(r)}{M}=V_t^2+V_p^2+\frac{2\phi(R)}{M}. \quad (3)$$

Here ϕ is the gravitational potential.

Then we have,

$$\rho_T(r)=\left(\frac{R}{r}\right)^2 4\pi\left(\frac{M}{2\pi kT}\right)^{3/2} \times\int\int\frac{e^{-(M/2kT)(V_p^2+V_t^2)}}{u_p}V_tV_p\,dV_t\,dV_p, \quad (4)$$

with

$$U_p=\left[V_p^2+\left(1-\frac{R^2}{r^2}\right)V_t^2-\frac{2\phi(r)-2\phi(R)}{M}\right]^{\frac{1}{2}}. \quad (5)$$

The domain of integration is determined by the condition

$$V_p^2+\left(1-\frac{R^2}{r^2}\right)V_t^2-\frac{2\phi(r)-2\phi(R)}{M}\geqslant 0.$$

Let

$$\frac{MV_t^2}{2kT}=x, \quad \frac{MV_p^2}{2kT}=y,$$

$$\frac{\phi}{kT}=\psi, \qquad \frac{R}{r}=\alpha.$$

Then

$$\rho_T(r)=N_0\alpha^2\pi^{-\frac{1}{2}}(I_1+I_2), \tag{6}$$

where

$$I_1=\int_0^{\psi(r)-\psi(R)} dy\int_{[\psi(r)-\psi(R)-y]/(1-\alpha^2)}^{\infty} dx \times\frac{e^{-(x+y)}}{[y+(1-\alpha^2)x-\psi(r)+\psi(R)]^{\frac{1}{2}}}, \tag{7}$$

$$I_2=\int_{\psi(r)-\psi(R)}^{\infty} dy\int_0^{\infty} dx \times\frac{e^{-(x+y)}}{[y+(1-\alpha^2)x-\psi(r)+\psi(R)]^{\frac{1}{2}}}. \tag{8}$$

In (7), let $Z=y+(1-\alpha^2)x-[\psi(r)-\psi(R)]$. After integration, we get

$$I_1=\frac{(1-\alpha^2)^{\frac{1}{2}}\pi^{\frac{1}{2}}}{\alpha^2}[e^{-\psi(r)+\psi(R)}-e^{-\{\psi(r)-\psi(R)\}/(1-\alpha^2)}]. \tag{9}$$

In (8) let $Z=[(1-\alpha^2)x+y-\psi(r)+\psi(R)]$ and

$$\theta=\tan^{-1}\left[\frac{y-\psi(r)+\psi(R)}{(1-\alpha^2)x}\right]^{\frac{1}{2}}. \tag{10}$$

After integration and simplification we have

$$I_2=\frac{\pi^{\frac{1}{2}}}{\alpha_2}\{e^{-[\psi(r)-\psi(R)]}-(1-\alpha^2)^{\frac{1}{2}}e^{-[\psi(r)-\psi(R)]}\}. \tag{11}$$

Substituting (9) and (11) into (7), we obtain a general expression for the density distribution of the ballistic component:

$$\rho_T(r)=N_0\{e^{-[\psi(r)-\psi(R)]} -(1-\alpha^2)^{\frac{1}{2}}e^{-[\psi(r)-\psi(R)]/(1-\alpha^2)}\}. \tag{12}$$

In terms of the gravitational constant G this may be rewritten as:

$$\rho_T(r)=N_0\{e^{-(1-\alpha)E}-(1-\alpha^2)^{1/2}e^{-E/(1+\alpha)}\} \tag{13}$$

where $E=GMm/RkT$.

Equation (13) corresponds to the integral equation (16) of Öpik and Singer (1961). It differs from the barometric formula

$$\rho_h(r)=N_0e^{-(1-\alpha)E} \tag{14}$$

by a term $N_o(1-\alpha^2)^{\frac{1}{2}}e^{-E/(1+\alpha)}$. This term is zero when $\alpha=1$ (i.e., $r=R$), it increases when α decreases, and its asymptotic value at $\alpha=0$ (i.e., $r=\infty$) is N_0e^{-E}.

The physical interpretation for this term can be explained by a study of the particle velocity distribution in the exosphere. From (2) and (3) we obtain

$$u_t^2-\frac{\alpha^2}{1-\alpha^2}u_p^2=\frac{\alpha^3}{1+\alpha}V_\infty^2-V_p^2, \tag{15}$$

or,

$$u_t^2-\frac{\alpha^2}{1-\alpha^2}u_p^2\leq\frac{\alpha^3}{1+\alpha}V_\infty^2, \tag{16}$$

where $V_\infty=(2KTE/M)^{\frac{1}{2}}$ is the escape velocity at the base of the exosphere.

This inequality (16) sets a limit on the velocity of ballistic particles. Hence we may classify the particles in the exosphere into five components according to their velocity, as shown in Fig. 1.

The particles whose velocities lie in regions (1) and (2) came from the base of the exosphere and are distributed as a Maxwellian. However, the particles in region (1) do not have enough energy to overcome the gravitational potential so they will return to earth after a trip in the exosphere. This is the re-entry component. The particles in region (2) are the escaping component.

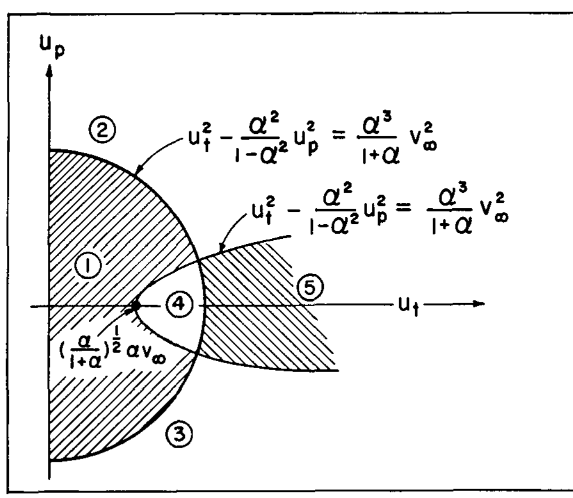

FIG. 1. Classes of exospheric components in velocity space: 1) ballistic re-entry, 2) ballistic escaping, 3) ballistic return-flux-of-escape, 4) bound-orbiting, 5) transient. As r increases (α decreases), the radius of the circle $((\alpha/1+\alpha)^{\frac{1}{2}}\alpha V_\infty)$ continues to decrease and the vertex of the hyperbola moves closer to the origin. The area of the transient component eventually covers the whole velocity space as $r\rightarrow\infty$.

Their total energy is greater than zero ($u^2 > u_\infty{}^2 = \alpha^2 V_\infty{}^2$), and they will therefore go to infinity. In the absence of collision and planetary plasma, these two are the only components found in the exosphere. In other words, the particles are confined in a region of the velocity space bounded by a hyperbola (equation 16) and a quarter circle ($u^2 = u_\infty{}^2$, $u_p < 0$).

The particles in region (3) are the return-flux-of-escape component. They came from infinity (outer space) and will fall into the base of the exosphere. Since the particles of these three components (1, 2, and 3) all possess ballistic trajectories in the exosphere, they may be called the ballistic component.

The particles in regions (4) and (5) do not intersect the base of the exosphere. They are created due to collisions of ballistic particles. The particles in region (4) are the bound orbiting component and circle around the planet. The particles in region (5) may be called the transient component. They pass through the exosphere and will go back to interplanetary space.

The barometric formula results from an isotropic Maxwellian velocity distribution; that is, all five regions in Fig. 1 are filled with particles distributed according to a Maxwellian law. This is true only if collisions are frequent enough. On the other hand, the formula (13) derived in this paper is based on the assumption that in the exosphere there are no collisions at all. Consequently, there cannot be particles in regions (4) and (5).

The density of the return-flux-of-escape component, denoted by $\rho_S(r)$, equals the density of the escaping component, which can be calculated by first deriving the escaping flux.

$$f_s = 4\pi R^2 N_0 \left(\frac{M}{2\pi kT}\right)^{\frac{3}{2}} \int_{\substack{|\vec{V}| > V_\infty \\ V_p > 0}} e^{-(MV^2/2kT)} V_p d\mathbf{V}$$

$$= 4\pi R^2 N_0 \left(\frac{kT}{2\pi M}\right)^{\frac{1}{2}} (HE) e^{-E}.$$

All the escaping particles come from the Maxwellian tail with average velocity very close to V_∞.

$$\overline{V_p{}^2} = \frac{\int_{V>V_\infty} e^{-(MV^2/2kT)} V_p{}^2 d\mathbf{V}}{\int_{V>V_\infty} e^{-(MV^2/2kT)} d\mathbf{v}} = \frac{H(E)}{3} V_\infty{}^2$$

$$\overline{v_t{}^2} = \frac{\int_{V>V_\infty} e^{-(MV^2/2kT)} V_t{}^2 d\mathbf{V}}{\int_{V>V_\infty} e^{-(MV^2/2kT)} d\mathbf{V}} = \frac{2H(E)}{3} V_\infty{}^2 \qquad (17)$$

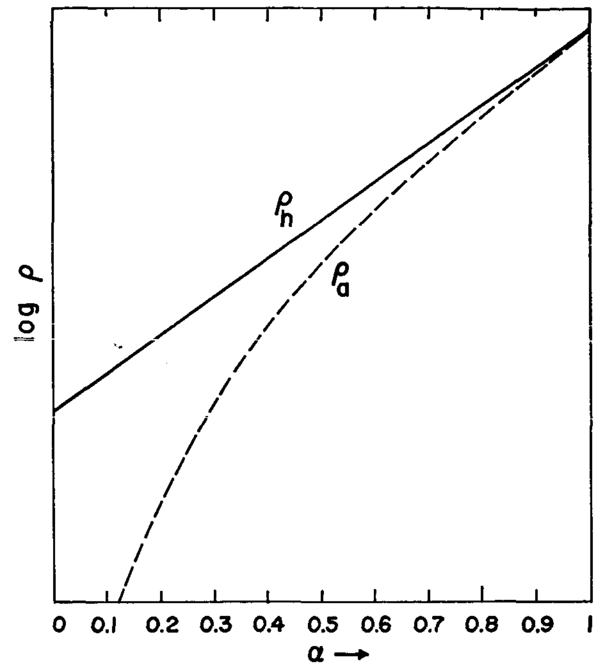

FIG. 2. Comparison of the barometric distribution and the distribution derived in this paper (equation 13).

where

$$H(E) = \frac{3}{2E} + \frac{1}{1 + \dfrac{\sqrt{\pi} e^E}{\alpha E^{\frac{1}{2}}}[1 - erf(E^{\frac{1}{2}})]} \cong 1 + \frac{2}{E} \text{ for large } E.$$

Therefore the density distribution of the return flux of escape is given by

$$\rho_s(r) = \frac{1}{4\pi r^2} \frac{f_s(r)}{\bar{u}_p}$$

$$= N_0 \frac{\alpha^2}{2\pi^{\frac{1}{2}}} \frac{(1+E)e^{-E}}{E^{\frac{1}{2}}\{H(E) - 1 + [1 - \frac{2}{3}H(E)\alpha]\alpha\}^{\frac{1}{2}}}. \qquad (18)$$

TABLE 1. Density distribution in an exosphere, according to the barometric model (ρ_h), Öpik-Singer (ρ_b), and the model given in this paper (ρ_t and ρ_a).

α	ρ_h	ρ_b	ρ_t	ρ_a
1	100.00	100.00	100.00	98.89
0.9	62.81	58.95	58.61	57.77
0.8	39.46	34.75	34.901	34.26
0.7	24.78	19.94	20.212	19.731
0.6	15.57	10.99	11.302	10.951
0.5	9.778	5.708	5.901	5.654
0.4	6.142	2.707	2.835	2.673
0.3	3.858	1.108	1.198	1.102
0.2	2.423	0.347	0.404	0.357
0.1	1.522	0.057	0.070	0.057
0	0.956	...	0.00	0.00

From (13) and (17), we have

$$\rho_a(r)=\rho_T(r)-\rho_s(r)$$

$$=N_0[e^{-(1-\alpha)E}-(1-\alpha^2)^{\frac{1}{2}}e^{-E/1+\alpha}]$$

$$-N_0\left[\frac{\alpha^2}{2\pi^{\frac{1}{2}}}\frac{(1+E)e^{-E}}{E^{\frac{1}{2}}\{H(E)-1+[1-\frac{2}{3}H(E)\alpha\}^{\frac{1}{2}}}\right]. \qquad (18)$$

The ρ_a here is the actual density distribution of a collisionless exosphere. It should agree with Öpik-Singer's numerical results.

In Table 1 and Fig. 2 we compare our formula with the barometric formula and Öpik-Singer's numerical results. The density distribution of a real exosphere should lay between the two curves, depending on the efficiency of collisions in the exosphere.

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REFERENCES

Öpik, E. J., and S. F. Singer, 1959: Distribution of density in a planetary exosphere. *Phys. Fluids*, 2, 653–655.

——, 1961: Distribution of density in a planetary exosphere II. *Phys. Fluids*, 4, 221–233.